Mechanics III

FIZIKA SJPO Training

28 March 2026

Contents

1 Notes

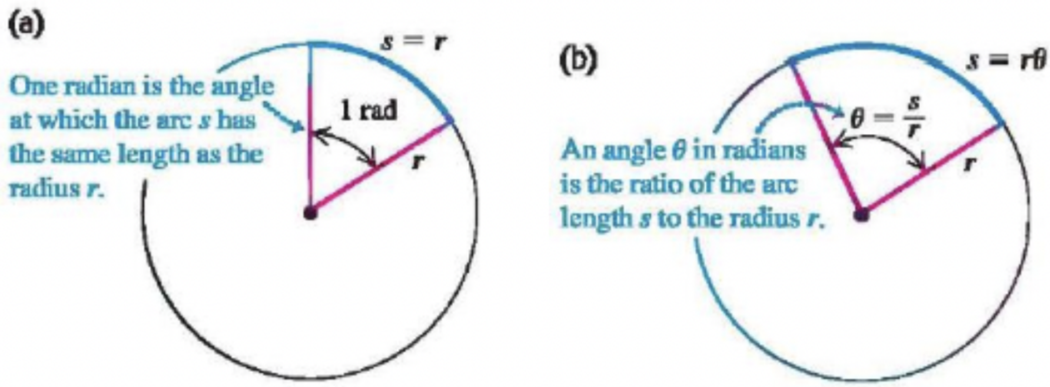
1.1 Circular Motion

Circular motion is yet another form of 2D motion. Generally, we are considering a point mass m moving in a circular trajectory of constant radius r. Because of the circular trajectory, the **velocity is constantly changing** (because its direction must always be changing).

1.1.1 Angular Quantities

We will be dealing with quantities relating to parts of a circle, i.e. **angular quantities.** The simplest one is the arc length s of a circle of radius r. If the angle spanned is θ, then:

$$s = r\theta \tag{1}$$



Remark. θ must be in **radians** for Equation (1) to work!

If we divide both sides of Equation (1) by time, noting that r is constant, we have:

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta \theta}{\Delta t} \quad \Longrightarrow \quad v = r\omega \tag{2}$$

where v is the linear speed and ω is the angular speed (units: rad/s).

If we further divide both sides of Equation (2) by time, we have:

$$\frac{\Delta v}{\Delta t} = r\frac{\Delta \omega}{\Delta t} \quad \Longrightarrow \quad a = r\alpha \tag{3}$$

where a is the linear (tangential) acceleration and α is the angular acceleration (units: rad/s^2).

We can quantify how long the circular motion takes by using the period T (the time taken to complete one full revolution). A closely related quantity is the frequency f (the number of revolutions per second), which is related to T by:

$$f = \frac{1}{T} \tag{4}$$

Since one revolution is $360° = 2\pi\,\text{rad}$, and this takes a time T, hence:

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{5}$$

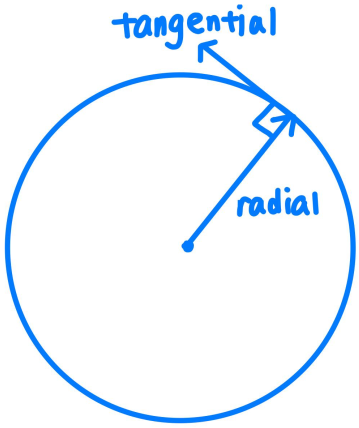
Example 1.1. Convert 30 rpm (revolutions per minute) to rad/s.

Solution. One revolution is equivalent to 2π rad and one minute has 60 s. Hence,

$$30\,\text{rpm} = 30 \times \frac{2\pi}{60}\,\text{rad/s} = \pi\,\text{rad/s}$$

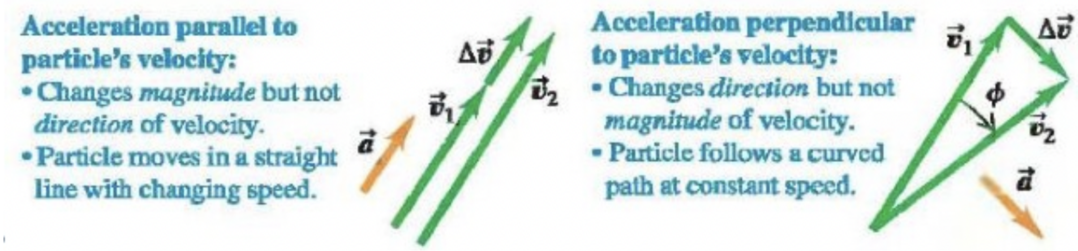
1.1.2 Radial & Tangential Components

Normally, the set of coordinate axes that we use are the x and y-axes. However, this is inconvenient for a circle. Instead, since it has symmetry in all directions extending out from the centre of the circle, we prefer to break things down into **radial** (along the radius) and **tangential** (along the tangent) components.



For an object moving in a uniform circle, its velocity must be **tangent** to the circle. We say that the radial velocity is 0.

Recall that an acceleration in a specific direction adds a $\Delta \boldsymbol{v}$ in that direction to the existing velocity vector. Hence, in this case, the tangential acceleration changes the magnitude of the velocity, while the radial acceleration changes the direction of the velocity.



1.1.3 Kinematics Of Uniform Circular Motion

In general, any object moving at constant speed v in a circle of radius r will experience a **centripetal acceleration** of magnitude a_c:

$$a_c = \frac{v^2}{r} \tag{6}$$

If we are given the angular speed ω instead, using Equation (2), we can rewrite the centripetal acceleration as:

$$a_c = r\omega^2 \tag{7}$$

This centripetal acceleration is directed **radially inwards, towards the centre of the circle.**

Remark. For uniform circular motion, even though the speed of the object is constant. take note that the **velocity is not constant!** This is because the direction of the velocity is continuously changing (so that the object stays in a circular path).

Example 1.2. A Ferris wheel has a radius of 15 m and completes 5 turns every minute. What is the acceleration (magnitude and direction) of a passenger at the lowest point during the ride?

Solution. The period of the Ferris wheel is $T = \frac{60}{5}\,\text{s} = 12\,\text{s}$. Then, we just simply apply Equations (5) and (7). We have:

$$a_\text{c} = r\omega^2 = r\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2\,(15)}{12^2} = 4.11\,\text{m/s}^2$$

The direction is vertically upwards (towards the centre of the circle), because the passenger is at the lowest point of the circle.

1.1.4 Kinematics Of Non-Uniform Circular Motion

In non-uniform circular motion, the object's speed is no longer constant! We can still define its centripetal acceleration as per Equations (6) and (7), but take note that because the speed is not constant, the centripetal acceleration will also not be constant. Instead, we prefer to analyse quantities at a specific instant in time.

This time, because speed is non-constant, there is a non-zero **tangential acceleration** of magnitude a_t, given by Equation (3). As such, the net acceleration must be obtained from a vector sum of the centripetal and tangential accelerations.

Example 1.3. A Ferris wheel with radius 14.0 m is turning anti-clockwise, about a horizontal axis through its centre. The linear speed of a passenger on its rim is equal to 7.00 m/s at some instant in time. If the revolution of the wheel speeds up at $1\,\text{rad/s}^2$, what is the acceleration (magnitude and direction) of a passenger at the lowest point of the ride?

Solution. The centripetal acceleration is given by Equation (6):

$$a_\text{c} = \frac{v^2}{r} = \frac{7.00^2}{14.0} = 3.50\,\text{m/s}^2$$

The tangential acceleration is given by Equation (3):

$$a_\text{t} = r\alpha = (14.0)\,(1) = 14.0\,\text{m/s}^2$$

Hence, the magnitude of the net acceleration is

$$a = \sqrt{{a_\text{c}}^2 + {a_\text{t}}^2} = \sqrt{3.50^2 + 14.0^2} = 14.4\,\text{m/s}^2$$

and the direction of the net acceleration is

$$\theta = \tan^{-1}\left(\frac{a_\text{c}}{a_\text{t}}\right) = \tan^{-1}\left(\frac{3.50}{14.0}\right) = 14.0^\circ \text{ North of East}$$

1.1.5 Dynamics Of Circular Motion

Since we have the existence of the centripetal acceleration, we can hence define the **centripetal force** with magnitude F_c as such:

$$F_\text{c} = ma_\text{c} = \frac{mv^2}{r} = mr\omega^2 \tag{8}$$

Likewise, this force also points radially inwards, towards the centre of the circle.

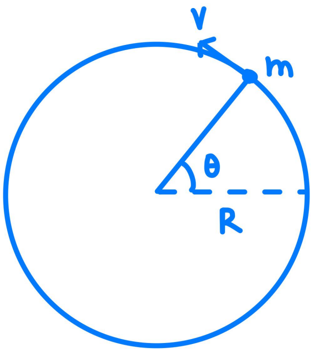
But, why does this force exist, and what is its significance? It turns out that the centripetal force is not a real force per se, but it is the **net force in the radial direction.**

Remark. Because the centripetal force is not a real force, it should **not** be included in FBDs! This is a very common trap.

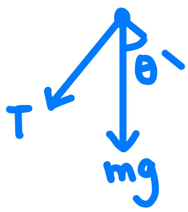
In general, you can follow this framework to solve circular motion questions:

1. Identify the centre and the plane of circular motion. (Don't skip this step! This is important because it tells us what is the radial direction we should consider for our centripetal force.)
2. Draw the FBD of the object, making sure not to include centripetal force in the diagram.
3. Find the net force pointing in the radially inwards direction on the object, and equate that to the centripetal force.

Example 1.4. A mass m is connected to a light, inextensible string, and it swings in a vertical circle of radius R, under the influence of gravity. Find the tension in the string at some angle θ from the horizontal, if the speed at that point is v, as per the diagram below.



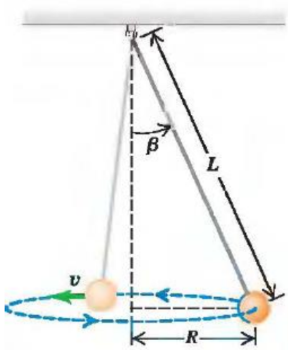
Solution. The FBD of the mass is as such:



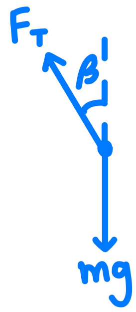
Equating the net radially inward force to the centripetal force, we have

$$T + mg\sin\theta = \frac{mv^2}{R} \quad \Longrightarrow \quad T = \frac{mv^2}{R} - mg\sin\theta$$

Example 1.5. A pendulum clock is constructed using a pendulum bob of mass m at the end of a light, inextensible string of length L. Instead of swinging back and forth, the bob moves in a horizontal circle with constant speed v, with the string making a constant angle β with the vertical. (a) Find the tension in the string. (b) Find the period of the pendulum.



Solution. (a) The FBD of the pendulum bob is as such (we use F_T for tension to not confuse it with the period later on):



Since the bob moves in a horizontal circle, the vertical forces must be balanced. Hence, we simply have:

$$F_T \cos\beta = mg \qquad \Longrightarrow \qquad F_T = \frac{mg}{\cos\beta}$$

(b) Now, let's look at the horizontal direction. From the figure, the radius of circular motion is $R = L\sin\beta$. Since the plane of circular motion is in this direction, we can equate the net radially inward force to the centripetal force as such:

$$F_T \sin\beta = \frac{mv^2}{R} = \frac{mv^2}{L\sin\beta} \qquad \Longrightarrow \qquad mg\tan\beta = \frac{mv^2}{L\sin\beta}$$

$$\Longrightarrow \qquad v = \sqrt{gL\sin\beta\tan\beta} = \sin\beta\sqrt{\frac{gL}{\cos\beta}}$$

Hence, the period is

$$T = \frac{2\pi R}{v} = \frac{2\pi L\sin\beta}{\sin\beta\sqrt{\frac{gL}{\cos\beta}}} = 2\pi\sqrt{\frac{L\cos\beta}{g}}$$

1.2 Gravitation

Now, we shall investigate gravity in more detail. We shall refer to the gravitational constant $G = 6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$ in many of our equations.

1.2.1 Force, Field, Potential & Potential Energy

The magnitude of the **gravitational force** $\boldsymbol{F_g}$ between two bodies of mass m_1 and m_2 separated by a distance r is:

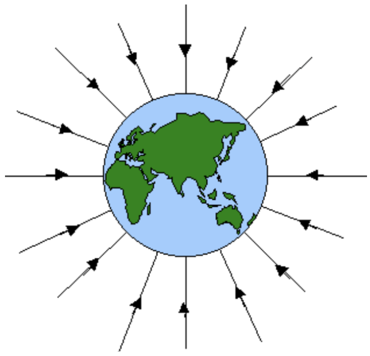
$$F_g = \frac{Gm_1m_2}{r^2} \tag{9}$$

and the direction is always attractive (towards each other).

The magnitude of the **gravitational field** $\boldsymbol{g}$ due to a mass m at a distance r away is:

$$g = \frac{Gm}{r^2} \tag{10}$$

and the direction is radially inwards, towards the mass.



From Equations (9) and (10), you can see that we can think of the gravitational field as the gravitational force per unit mass.

The **gravitational potential energy** U_g associated with the interaction of two bodies of mass m_1 and m_2 separated by a distance r is:

$$U_g = -\frac{Gm_1m_2}{r} \tag{11}$$

The **gravitational potential** ϕ_g due to a mass m at a distance r away is:

$$\phi_g = -\frac{Gm}{r} \tag{12}$$

From Equation (11) and (12), you can see that we can think of the gravitational potential as the gravitational potential energy per unit mass.

Remark. Take note that $\boldsymbol{F}_g$ and $\boldsymbol{g}$ are vectors, while U_g and ϕ_g are scalars. Hence, especially when adding gravitational forces and gravitational fields, remember that we have to add them as vectors!

Remark. The **negative signs** in Equations (11) and (12) are extremely important! Do not forget about them!

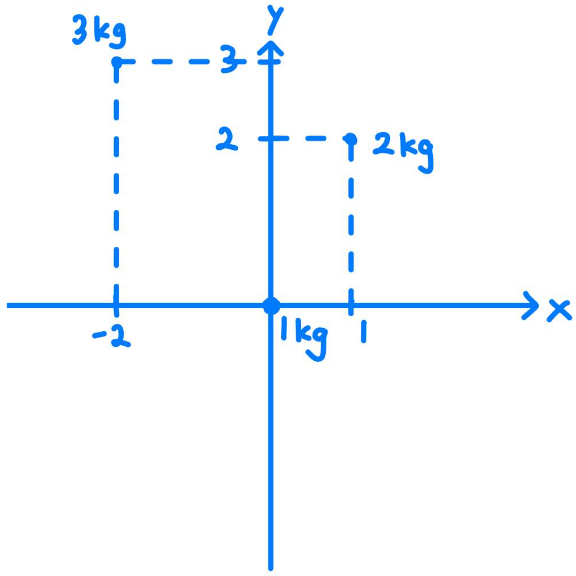
Now, what other relationships can we form between the four quantities above? It turns out that, we also have:

1. The negative of the gradient of the graph of U_g against r is F_g.
2. The negative of the gradient of the graph of ϕ_g against r is g.
3. The negative of the area under the graph of F_g against r is U_g.
4. The negative of the area under the graph of g against r is ϕ_g.

Once again, you are reminded of the sheer importance of the **negative signs!**

Example 1.6. Consider a typical x-y coordinate system, with lengths given in m. Three masses are placed at different locations. A 1 kg mass is placed at $(0, 0)$, a 2 kg mass is placed at $(1, 2)$, and a 3 kg mass is placed at $(-2, 3)$. (a) Find the gravitational field (magnitude and direction) at $(0, 0)$. (b) Find the gravitational force (magnitude and direction) on the 1 kg mass. (c) Find the gravitational potential at $(0, 0)$. (d) Find the total gravitational potential energy of the system.

Solution. We shall let m_1, m_2 and m_3 represent the respective masses, and r_{12}, r_{23} and r_{13} represent the distances between the respective masses. Let's first draw a diagram of the masses:



(a) At $(0,0)$, the gravitational field is due to the 2 kg and 3 kg masses:

$$\boldsymbol{g}(0,0) = \frac{Gm_2}{{r_{12}}^2}\begin{pmatrix}\cos\theta_{12}\\ \sin\theta_{12}\end{pmatrix} + \frac{Gm_3}{{r_{13}}^2}\begin{pmatrix}-\cos\theta_{13}\\ \sin\theta_{13}\end{pmatrix}$$

$$= \frac{(6.67\times10^{-11})(2)}{1^2+2^2}\begin{pmatrix}\frac{1}{\sqrt{1^2+2^2}}\\ \frac{2}{\sqrt{1^2+2^2}}\end{pmatrix} + \frac{(6.67\times10^{-11})(3)}{2^2+3^2}\begin{pmatrix}-\frac{2}{\sqrt{2^2+3^2}}\\ \frac{3}{\sqrt{2^2+3^2}}\end{pmatrix} = \begin{pmatrix}3.39\times10^{-12}\\ 3.67\times10^{-11}\end{pmatrix}\ \text{N/kg}$$

(b) Since the gravitational field is the gravitational force per unit mass, the gravitational force on the 1 kg mass is just

$$\boldsymbol{F}_g = m_1\boldsymbol{g}(0,0) = (1)\begin{pmatrix}3.39\times10^{-12}\\ 3.67\times10^{-11}\end{pmatrix} = \begin{pmatrix}3.39\times10^{-12}\\ 3.67\times10^{-11}\end{pmatrix}\ \text{N}$$

(c) At $(0,0)$, the gravitational potential is due to the 2 kg and 3 kg masses:

$$\phi_g(0,0) = -\frac{Gm_2}{r_{12}} - \frac{Gm_3}{r_{13}} = -\frac{(6.67\times10^{-11})(2)}{\sqrt{1^2+2^2}} - \frac{(6.67\times10^{-11})(3)}{\sqrt{2^2+3^2}} = -1.15\times10^{-10}\ \text{J/kg}$$

(d) This one is the tricky part. Remember that gravitational potential energy arises due to the interaction between two masses. Hence, we need to be careful to **not double-count** interactions!

Between three masses, there are three interactions. Hence, the total gravitational potential energy is just

$$U_g = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_1m_3}{r_{13}}$$

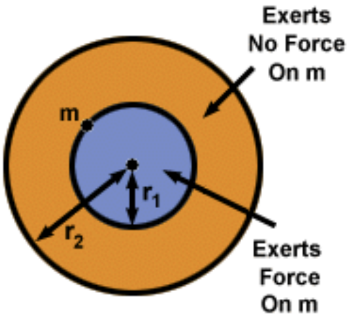
$$= -\frac{(6.67\times10^{-11})(1)(2)}{\sqrt{1^2+2^2}} - \frac{(6.67\times10^{-11})(2)(3)}{\sqrt{1^2+3^2}} - \frac{(6.67\times10^{-11})(1)(3)}{\sqrt{2^2+3^2}} = -2.42\times10^{-10}\ \text{J}$$

1.2.2 Gravitational Field Inside A Spherical Mass

When we are inside any mass with **spherical symmetry,** certain rules for finding the gravitational field apply.

The first would be **Newton's Shell Theorem,** which states that **outside** of a uniform spherical mass (regardless of whether it is a hollow or solid sphere), the gravitational field would be the same as if all the mass were concentrated at its **centre.**

The second would be the concept of **effective mass,** m_{eff}. This effective mass is the mass of the largest sphere that you can draw, for which the point of consideration for the gravitational field is just outside that largest sphere. When we are calculating the gravitational field, we will use m_{eff} as the mass producing the field.



Essentially, the effective mass is the mass of the blue/purple region, and this is the mass that contributes to the gravitational field. The orange region is outside and hence contributes no gravitational field.

For a **hollow shell** (i.e. all the mass is uniformly distributed on the surface) of mass M and radius R, we can consider two cases for the distance r from the centre of the shell.

1. **Inside the shell** (when $r < R$): $g = 0$
2. **Outside the shell** (when $r \geq R$): $g = \frac{GM}{r^2}$, towards the centre of the shell

Example 1.7. For a **solid shell** (i.e. all the mass is uniformly distributed in the volume) of mass M and radius R, we can also consider the same two cases for the distance r from the centre of the shell. Find the magnitude of the gravitational field for both cases.

Solution. The case for **outside the shell** (when $r \geq R$) is easier. In this case, the effective mass $m_{\text{eff}} = M$, because the largest sphere we can draw contains the entire mass. Hence, $g = \frac{Gm_{\text{eff}}}{r^2} = \frac{GM}{r^2}$ for this case.

The case for **inside the shell** (when $r < R$) is slightly harder. In this case, the effective mass is the mass of the sphere of radius r. By taking the ratio of the volumes (which is the same as the ratio of the masses, due to the uniform density), we have $m_{\text{eff}} = M\frac{r^3}{R^3}$, and hence, $g = \frac{Gm_{\text{eff}}}{r^2} = \frac{GMr}{R^3}$ in this case.

In any case, the gravitational field is directed towards the centre of the sphere.

Example 1.8. Many gravitation questions link back to circular motion. Suppose that Earth is now a hollow sphere of mass M and radius R, and suppose that you (of mass m) are standing on the interior surface of the sphere. Find the angular speed at which the "Earth" must rotate about its centre to create an "artificial gravity" of g at your position.

Solution. The "Earth" itself now contributes no gravitational field, because we are inside a hollow shell. Instead, this "artificial gravity" originates from centripetal acceleration!

The only force acting on you would be a normal force N directed towards the centre of the "Earth". Since this is the only force, the centripetal force must provide for this force:

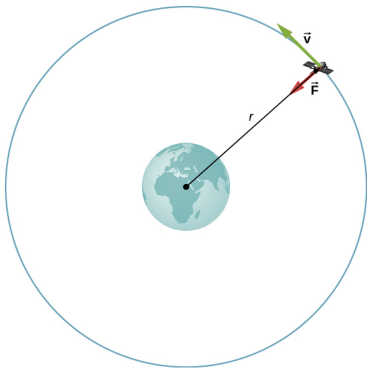
$$N = mR\omega^2 = mg_{\text{eff}}$$

where $g_{\text{eff}} = g$. Hence,

$$g = R\omega^2 \quad \Longrightarrow \quad \omega = \sqrt{\frac{g}{R}}$$

1.2.3 Satellites In Circular Orbit

Consider a satellite of mass m flying in a circular orbit of radius R around a planet of mass M.



The only force acting on the satellite is the gravitational force. Since the satellite is in a circular orbit, the gravitational force is hence equal to the centripetal force:

$$\frac{GMm}{R^2} = \frac{mv^2}{R} \quad \Longrightarrow \quad v = \sqrt{\frac{GM}{R}} \tag{13}$$

Equation (13) tells us the speed of the satellite.

Remark. Observe that the speed of the satellite is independent of its own mass m! Instead, it depends on the planet's mass M.

We can hence also find the angular speed of the satellite:

$$\omega = \frac{v}{R} = \sqrt{\frac{GM}{R^3}} \tag{14}$$

We can hence also find the period of the satellite:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{15}$$

Equation (15) is also referred to as **Kepler's 3rd Law.**

We can also analyse the energy of the satellite, which will consist of kinetic energy and gravitational potential energy. Often, we are also interested in the **total energy,** which is just the sum of the two:

$$E = K + U_g = \frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{GMm}{2R} - \frac{GMm}{R} = -\frac{GMm}{2R} \tag{16}$$

Here, we have used Equation (13) to simplify the kinetic energy term.

From Equation (16), you may also observe the following useful relationships, that hold **everywhere** along a **circular orbit:**

$$T = \frac{1}{2}U_g \tag{17}$$

$$K = -\frac{1}{2}U_g \tag{18}$$

$$T = -K \tag{19}$$

Remark. Equations (17) to (19) only hold for **circular orbits!** It will not hold for other types of orbits, such as elliptical orbits, even though we are not covering them.

If no forces other than the gravitational force act on the satellite, then the total energy of the satellite remains constant (COE). Otherwise, if other external forces act on the satellite, the total energy, and hence the radius of the circular orbit as per Equation (16), will change!

Example 1.9. Suppose a satellite of mass m orbiting at a distance d from Earth (of mass M)'s centre is meant to drop its orbit to a distance of $\frac{d}{2}$ from Earth's centre. Both orbits are circular. (a) How much work is required? (b) What is the ratio of the satellite's new speed to its old speed?

Solution. (a) The work required is just the difference in the total energy between the two orbits:

$$W = \Delta E = -\frac{GMm}{2R_\mathrm{f}} - \left(-\frac{GMm}{2R_\mathrm{i}}\right) = \frac{GMm}{2}\left(\frac{1}{R_\mathrm{i}} - \frac{1}{R_\mathrm{f}}\right) = \frac{GMm}{2}\left(\frac{1}{d} - \frac{1}{\frac{d}{2}}\right) = -\frac{GMm}{2d}$$

The negative work required actually means that the satellite must "brake" initially to slow down for gravity to pull it inward first.

(b) From Equation (13), $v \propto \frac{1}{\sqrt{R}}$. Hence,

$$\frac{v_\mathrm{f}}{v_\mathrm{i}} = \sqrt{\frac{R_\mathrm{i}}{R_\mathrm{f}}} = \sqrt{2}$$

The increased new speed may appear counter-intuitive, because we established that the satellite must "brake" first to slow down, so that it can be pulled inwards. However, it loses so much gravitational potential energy by moving inwards, such that its kinetic energy (and speed) actually increases in its new orbit!

1.2.4 Escape Velocity

The **escape velocity** v_esc is the minimum speed required to launch an object so that it completely escapes the gravitational influence of some other object.

Consider launching some object of mass m from the surface of a planet of mass M and radius R. For it to just barely escape the planet's gravitational influence, the final gravitational potential energy must be 0. And, since we are looking for the minimum possible initial speed, we should also set the final speed (and hence final kinetic energy) to be 0. Hence,

$$\frac{1}{2}m{v_\mathrm{esc}}^2 - \frac{GMm}{R} = 0 \quad \Longrightarrow \quad v_\mathrm{esc} = \sqrt{\frac{2GM}{R}} \tag{20}$$

Equation (20) gives the typical escape velocity of an object.

Remark. Observe that the escape velocity of an object is independent of its own mass m! Instead, it depends on the planet's mass M.

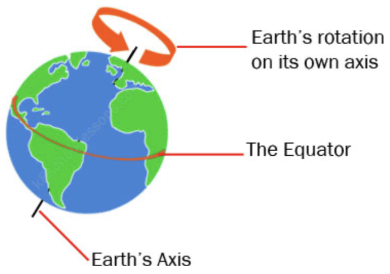
In a more general situation, what you want to do is to just set the final total energy to be 0 to find the escape velocity.

Example 1.10. Consider Earth, which can be treated as a solid sphere of mass $M_\mathrm{E} = 5.97 \times 10^{24}\,\mathrm{kg}$ and radius $R = 6.38 \times 10^6\,\mathrm{m}$. (a) Calculate the escape velocity, ignoring the rotation of the Earth about its own axis. (b) Now, suppose that the rotation of the Earth about its own axis cannot be ignored. Find the minimum amount of velocity we need to give an object at the Equator for it to escape.

Solution. (a) By simply putting the numbers into Equation (20), we obtain:

$$v_{\text{esc}} = \sqrt{\frac{2\,(6.67 \times 10^{-11})\,(5.97 \times 10^{24})}{6.38 \times 10^{6}}} = 11.2\,\text{km/s}$$

(b) The rotation of the Earth about its own axis produces some tangential speed at the surface of the Earth, which helps reduce the velocity we need to give an object, if we propel it in the same direction as this tangential speed.



The period of Earth's rotation about its own axis is $T = 24\,\text{h}$ (this is what gives rise to day and night cycles). Hence,

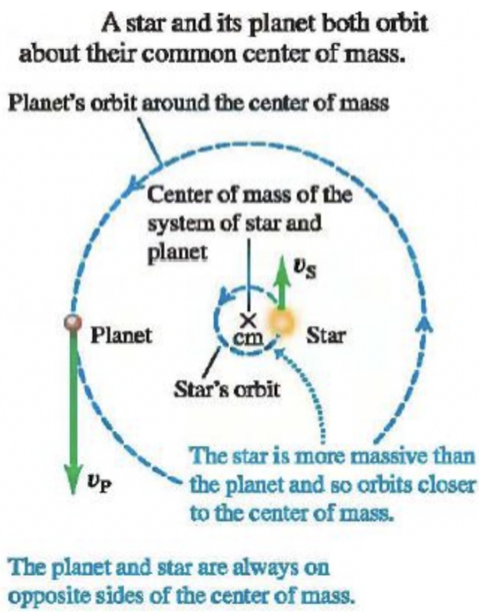
$$v_{\text{rot}} = \frac{2\pi R}{T} = \frac{2\pi\,(6.38 \times 10^{6})}{24 \times 60 \times 60} = 464\,\text{m/s}$$

Hence, the new velocity we need to give is :

$$v'_{\text{esc}} = v_{\text{esc}} - v_{\text{rot}} = 10.7\,\text{km/h}$$

1.2.5 Binary Star Systems

In reality, when two bodies interact via a gravitational force and are orbiting, they will orbit **about their CM,** as per the diagram below.



In most of the cases we are dealing with, an approximation we make is that $m \ll M$ (i.e. the mass of the satellite is much smaller than the mass of the planet). This assumption means that the CM is extremely near (and hence approximately at) the planet, hence we can **assume the planet is stationary** and the satellite is orbiting about a stationary planet.

However, when we talk about two bodies of **similar mass** orbiting around each other (for example, two planets), this approximation may **no longer be justifiable!** In this case, we need to fall back to finding the common point of orbit (the CM of the two bodies).

Example 1.11. Consider two planets A and B (separated by a distance d) orbiting around each other, with masses such that $M_\mathrm{A} = 4M_\mathrm{B}$. (a) What is the ratio of their angular speeds? (b) What is the ratio of their linear speeds?

Solution. (a) You can instantly conclude that $\frac{\omega_\mathrm{A}}{\omega_\mathrm{B}} = 1$. A very quick argument is as such: Suppose that the ratio is not 1 (i.e. $\omega_\mathrm{A} \neq \omega_\mathrm{B}$). Then, this means that when A makes a full orbit, B would not have made a full orbit/would have exceeded a full orbit. A and B hence no longer line up with each other, and hence will never return to their initial, starting configuration, hence they will not be orbiting each other if this is the case!

In general, for any binary star system, the **angular speeds are equal.**

(b) We can first find the position of the CM. Let us measure it with respect to planet A. Hence, the CM is located at $\frac{M_\mathrm{B} d}{M_\mathrm{A}+M_\mathrm{B}} = \frac{d}{5}$ away from planet A, and $\frac{4d}{5}$ away from planet B. This also gives the respective radii of orbit for planets A and B.

Now, by equating the gravitational force to the centripetal force for both, we have:

$$\text{Planet A: } \frac{GM_\mathrm{A}M_\mathrm{B}}{d^2} = \frac{M_\mathrm{A}{v_\mathrm{A}}^2}{\frac{d}{5}} \implies v_\mathrm{A} = \sqrt{\frac{GM_\mathrm{B}}{5d}}$$

$$\text{Planet B: } \frac{GM_\mathrm{A}M_\mathrm{B}}{d^2} = \frac{M_\mathrm{B}{v_\mathrm{B}}^2}{\frac{4d}{5}} \implies v_\mathrm{B} = \sqrt{\frac{4GM_\mathrm{A}}{5d}} = 4\sqrt{\frac{GM_\mathrm{B}}{5d}}$$

Hence, the ratio is $\frac{v_\mathrm{A}}{v_\mathrm{B}} = \frac{1}{4}$.

2 Problems

Problem 2.1 (SJPO 2008)**.** A particle moves at a constant speed in a circular path of radius 2.06 cm. If the particle makes 4 revolutions per second, what is the magnitude of its acceleration?

(A) $20\,\mathrm{m/s^2}$

(B) $18\,\mathrm{m/s^2}$

(C) $13\,\mathrm{m/s^2}$

(D) $15\,\mathrm{m/s^2}$

(E) $24\,\mathrm{m/s^2}$

Solution. (C)

Problem 2.2 (SJPO 2011). A 1.3 kg tether ball is on a 2.1 m string which makes an angle of 22° to the vertical as it moves around the pole in a horizontal plane. What is its speed?

(A) 1.4 m/s

(B) 1.6 m/s

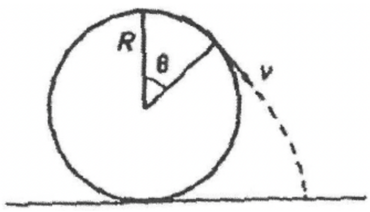
(C) 1.8 m/s

(D) 2.0 m/s

(E) 2.2 m/s

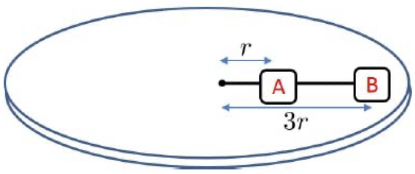
Solution. (D)

Problem 2.3. A small bead of mass m is at rest at the top of a fixed circle of radius R. Due to a small perturbation, it slips down the circular path without friction. Determine the angle θ (as per the diagram below) at which the bead falls off the surface of the circle.



Solution. 48.2°

Problem 2.4 (SJPO 2015). Two identical masses, A and B, are tied to strings and placed on a horizontal frictionless disc as in the figure below. The two masses are then set to move about the centre of the disc with the same angular velocity ω. Given that the tension of the string connecting mass A to the centre of the disc is T, determine the tension of the string connecting mass B to mass A.



(A) $\frac{1}{4}T$

(B) $\frac{3}{4}T$

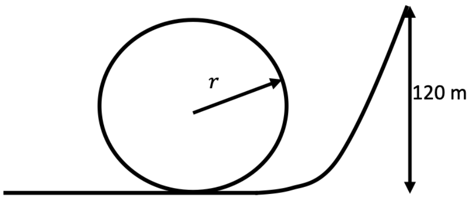
(C) T

(D) $3T$

(E) $4T$

Solution. (B)

Problem 2.5 (SJPO 2016). A 360 kg roller coaster car is initially at rest at a height of 120 m above the ground. It goes to the ground and does a circular loop of radius r. Assume that friction and energy losses are negligible, the car is small and is not attached to the track. What is the maximum radius r so that the roller coaster does not leave the track?



(A) 120 m

(B) 60 m

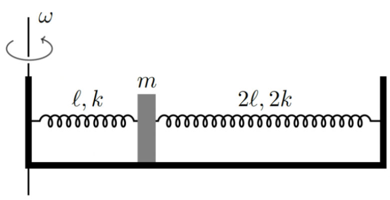
(C) 48 m

(D) 42 m

(E) 36 m

Solution. (C)

Problem 2.6 ($F = ma$ 2024). A block of mass m is connected to the walls of a frictionless box by two massless springs with relaxed lengths l and $2l$, and spring constants k and $2k$ respectively. The length of the box is $3l$. The system rotates with a constant angular velocity ω about one of its walls. Suppose the block stays at a constant distance r from the axis of rotation, without touching either of the walls. Determine an expression for r.



(A) $\frac{2kl}{2k-m\omega^2}$

(B) $\frac{2kl}{2k+m\omega^2}$

(C) $\frac{2kl}{3k+m\omega^2}$

(D) $\frac{3kl}{3k-m\omega^2}$

(E) $\frac{3kl}{3k+m\omega^2}$

Solution. (D)

Problem 2.7 ($F = ma$ 2008). A uniform circular ring of radius R is fixed in place. A particle is placed on the axis of the ring at a distance much greater than R and allowed to fall towards the ring under the influence of the ring's gravity. The particle achieves a maximum speed v. The ring is replaced with one of the same linear mass density but radius $2R$, and the experiment is repeated. What is the new maximum speed of the particle?

(A) $\frac{1}{2}v$

(B) $\frac{1}{\sqrt{2}}v$

(C) v

(D) $\sqrt{2}v$

(E) $2v$

Solution. (D)

Problem 2.8 (SJPO 2018). Approximately what height (in terms of Earth radii) from the Earth's surface is the height of a circular geosynchronous orbit (24 h period)? Take $M_{\mathrm{E}} = 5.97 \times 10^{24}\,\mathrm{kg}$ and $R_{\mathrm{E}} = 6.38 \times 10^{6}\,\mathrm{m}$.

(A) 3

(B) 4

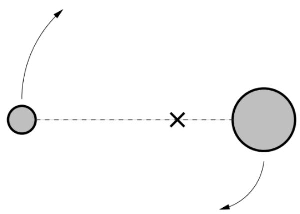
(C) 5

(D) 6

(E) 7

Solution. (D) *From earth's surface

Problem 2.9 ($F = ma$ 2009). Two stars of masses M and $3M$ orbit their common centre of mass as shown in the diagram below. The distance between the two stars is d. Determine the period of orbit for the star of mass $3M$.



(A) $\pi\sqrt{\frac{d^3}{GM}}$

(B) $\frac{3\pi}{4}\sqrt{\frac{d^3}{GM}}$

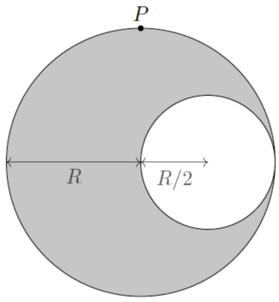
(C) $\pi\sqrt{\frac{d^3}{3GM}}$

(D) $2\pi\sqrt{\frac{d^3}{GM}}$

(E) $\frac{\pi}{4}\sqrt{\frac{d^3}{GM}}$

Solution. (A)

Problem 2.10 ($F = ma$ 2021). A spherical cavity of radius $\frac{R}{2}$ is dug out of a spherical planet with uniform mass density of mass M and radius R. What is the magnitude of the gravitational field at point P in the diagram below? *Hint: The cavity can be treated as a negative mass.*



(A) $0.200\frac{GM}{R^2}$

(B) $0.457\frac{GM}{R^2}$

(C) $0.829\frac{GM}{R^2}$

(D) $0.900\frac{GM}{R^2}$

(E) $0.912\frac{GM}{R^2}$

Solution. (E)

Problem 2.11 ($F = ma$ 2022). Two identical spherically symmetric planets, each of mass M, are somehow held at rest with respect to each other. Each planet has radius R, and the distance between the centres of the planets is $4R$. If a rocket is launched from the surface of one planet with speed v, what is the minimum speed v so that the rocket can reach the other planet?

(A) $\sqrt{\frac{2GM}{R}}$

(B) $\sqrt{\frac{GM}{R}}$

(C) $\sqrt{\frac{3GM}{4R}}$

(D) $\sqrt{\frac{2GM}{3R}}$

(E) $\sqrt{\frac{GM}{2R}}$

Solution. (D)

Problem 2.12 ($F = ma$ 2024). A spherical shell is made from a thin sheet of material with a mass per area of σ. Consider two points, P_1 and P_2, which are close to each other, but just inside and outside the sphere respectively. If the accelerations due to gravity at these points are $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ respectively, what is the value of $|\boldsymbol{g}_1 - \boldsymbol{g}_2|$?

(A) $\pi G\sigma$

(B) $\frac{4}{3}\pi G\sigma$

(C) $2\pi G\sigma$

(D) $4\pi G\sigma$

(E) $8\pi G\sigma$

Solution. (D)

Problem 2.13 ($F = ma$ 2014). A spherical cloud of dust in space has a uniform density ρ_0 and a radius R_0. The gravitational acceleration of free fall at the surface of the cloud due to the mass of the cloud is g_0. A process (heat expansion) causes the cloud to suddenly grow to a radius $2R_0$, while maintaining a uniform (but not constant) density. What is the gravitational acceleration of free fall at a point R_0 away from the centre of the cloud due to the mass of the cloud now?

(A) $\frac{1}{32}g_0$

(B) $\frac{1}{16}g_0$

(C) $\frac{1}{8}g_0$

(D) $\frac{1}{4}g_0$

(E) $\frac{1}{2}g_0$

Solution. (C)

Problem 2.14 ($F = ma$ 2019). A spherical cloud of dust has uniform mass density ρ and radius R. Satellite A is orbiting the cloud at its edge, in a circular orbit of radius R, and satellite B is orbiting just inside the cloud, in a circular orbit of radius r, with $r < R$. Which of the following statements about the periods T and speeds v of the satellites is true? Neglect drag.

(A) $T_A > T_B$ and $v_A > v_B$

(B) $T_A > T_B$ and $v_A < v_B$

(C) $T_A < T_B$ and $v_A > v_B$

(D) $T_A < T_B$ and $v_A < v_B$

(E) $T_A = T_B$ and $v_A > v_B$

Solution. (E)